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AFDELING NUMERIEKE WISKUNDE
(DEPARTMENT OF NUMERICAL MATHEMATICS)

NW 151/83 APRIL

P.J. VAN DER HOUWEN & H.J.J. TE RIELE

LINEAR MULTISTEP METHODS FOR VOLTERRA INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Preprint

amsterdam 1983

stichting
mathematisch
centrum

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kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 65R20

Linear multistep methods for Volterra integral and integro-differential equations *)

by

P.J. van der Houwen & H.J.J. te Riele

ABSTRACT

A general class of linear multistep methods is presented for numerically solving first and second kind Volterra integral equations, and Volterra integro-differential equations. These so-called *VLM methods*, which include the well-known direct quadrature methods, allow for a unified treatment of the problems of consistency and convergence, and have a pendant in linear multistep methods for ODEs, as treated in any textbook on computational methods in ordinary differential equations.

General consistency and convergence results are presented (and proved in an Appendix), together with results of numerical experiments which support the theory.

KEY WORDS & PHRASES: *numerical analysis; Volterra integral and integro-differential equations; linear multistep methods; consistency; convergence*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

We consider *Volterra integral equations* of the form

$$(1.1)\qquad \theta y(t) = g(t) + \int_{t_0}^{t} K(t,\tau,y(\tau))d\tau,\quad t \in I := [t_0,T],\ \theta = 0,1.$$

This equation is called of the *first kind* if $\theta = 0$ and of the *second kind* if $\theta = 1$. Furthermore, we consider *Volterra integro-differential* equations

$$(1.2)\qquad \frac{dy}{dt} = f(t,y(t),z(t)),\quad z(t) = g(t) + \int_{t_0}^{t} K(t,\tau,y(\tau))d\tau,\quad t \in I,$$

where $y(t_0) = y_0$. In these equations $y(t)$ is the unknown function and g, K and f are given, nonsingular functions on I, $S \times \mathbb{R}$ and $I \times \mathbb{R} \times \mathbb{R}$, respectively, where $S := \{(t,\tau),\ t_0 \le \tau \le t \le T\}$. In order to ensure the existence of a unique, continuous solution of (1.1) and (1.2), the following conditions should be satisfied, respectively:

Conditions for the existence of a unique solution $y(t) \in C(I)$ *of* (1.1) *with* $\theta = 1$

- $K(t,\tau,y)$ is continuous with respect to t and τ, for all $(t,\tau) \in S$;
- K satisfies a (uniform) Lipschitz condition with respect to y, i.e., $|K(t,\tau,y) - K(t,\tau,z)| \le L_1 |y-z|$, for all $(t,\tau) \in S$, for all finite $y,z \in \mathbb{R}$;
- $g(t) \in C(I)$. □

Conditions for the existence of a unique solution $y(t) \in C(I)$ *of* (1.1) *with* $\theta = 0$

- $K(t,\tau,y) \in C^1(S\times\mathbb{R})$;
- for $t = \tau$ the derivative $\partial K/\partial y$ is bounded away from zero: $|\partial K(t,t,y)/\partial y| \ge r_0 > 0$ for all $t \in I$, $y \in \mathbb{R}$;
- $K(t,\tau,y)$ satisfies a (uniform) Lipschitz condition with respect to y on $S \times \mathbb{R}$;
- $g(t) \in C^1(I)$ with $g(t_0) = 0$. □

Conditions for the existence of a unique solution $y(t) \in C^1(I)$ *of* (1.2), *for given initial value* $y(t_0) = y_0$

The following three (uniform) Lipschitz conditions:

- $|f(t,y_1,z) - f(t,y_2,z)| \le L_1 |y_1 - y_2|$, for all $t \in I$, for all finite $z, y_1, y_2 \in \mathbb{R}$;
- $|f(t,y,z_1) - f(t,y,z_2)| \le L_2 |z_1 - z_2|$, for all $t \in I$, for all finite $y, z_1, z_2 \in \mathbb{R}$;
- $|K(t,\tau,y_1) - K(t,\tau,y_2)| \le L_3 |y_1 - y_2|$, for all $(t,\tau) \in S$, for all finite $y_1, y_2 \in \mathbb{R}$. □

A common, simple way of solving (1.1) numerically is obtained by writing these equations down in a sequence of equidistant points

$$(1.3) \qquad t_n := t_0 + nh, \quad n = 0(1)N, \quad h \text{ fixed and } t_N = T,$$

approximating the integral term by some suitably chosen quadrature formula, and solving the resulting equation for $y(t_n)$, successively for $n = n_0(1)N$, where n_0 is some suitable starting index. Equation (1.2) is commonly solved by integrating the differential equation in the points (1.3) (say), using an LM formula for ODEs, thereby approximating $z(t_j)$ with some suitably chosen quadrature formula. All these methods are called *linear multistep* (LM) *direct quadrature* (DQ) methods. DQ methods may give satisfactory results, but sometimes the results with DQ are completely worthless as was demonstrated for first kind equations by LINZ [9, p. 67], where he applied a fourth order Gregory quadrature method to the very simple integral equation

$$(1.4) \qquad 0 = -\sin t + \int_0^t \cos(t-\tau)y(\tau)d\tau, \; I = [0,2], \text{ with exact solution } y(t) = 1.$$

The "approximate" values obtained for $y(2)$ were 8.4 and 1.5×10^7 for $h = 0.1$ and $h = 0.05$, respectively. For second kind equations too the Gregory rules will fail if large Lipschitz constants for the kernel function with respect to y are involved.

In this paper we present a general class of linear multistep methods for (1.1) and (1.2) which includes the DQ methods. (It should be remarked that such methods for *second kind Volterra integral* equations were already introduced in [6] and results were presented without proof). A characteristic feature of this class is that it involves linear combinations, not only of y - and K - values, but also of values of the auxiliary function (called the *lag* term)

$$(1.5) \qquad Y(t,s) := g(t) + \int_{t_0}^{s} K(t,\tau,y(\tau))d\tau,$$

for $(t,s) \in S$. Note that we may write (1.1) as $\theta y(t) = Y(t,t)$.

This general class will be called *Volterra linear multistep* (VLM) methods. VLM methods allow for a uniform treatment of the problems of consistency and convergence, and have a pendant in linear multistep methods for *ordinary differential equations*, as treated, e.g., by LAMBERT in [8].

In Section 2 of this paper we treat VLM methods for Volterra integral equations of the second and of the first kind jointly. Numerical experiments with several examples of VLM methods are reported. In a similar way as is done in Section 2, Section 3 treats VLM methods for integro-differential equations. It turns out that several results of Section 2 for *second* kind Volterra integral equations can be used in Section 3. The proofs of the theorems presented in Sections 2 and 3 are given, as far as they are non-trival, in an Appendix to this paper.

2. VLM METHODS FOR VOLTERRA INTEGRAL EQUATIONS

2.1. The general VLM method

In order to state our general VLM method for (1.1) we introduce numerical approximations y_n to $y(t_n)$ and $Y_n(t)$ to $Y(t,t_n)$, and we let

$$(2.1.1) \qquad K_n(t) := K(t,t_n,y_n), \quad n \geq 0.$$

We assume that $Y_n(t)$, $t \geq t_n$, will be computed by a quadrature formula of the form

$$(2.1.2) \qquad Y_n(t) := g(t) + h \sum_{j=0}^{n} w_{nj} K_j(t), \quad n \geq n_1,$$

where the w_{nj} are given weights and n_1 is sufficiently large in order to ensure the required order of accuracy. When we say that the *order of this quadrature formula* is r, we mean that for any $t \geq t_n$

$$(2.1.3)\qquad E_n(h;t) := \int_{t_0}^{t_n} K(t,\tau,y(t))d\tau - h\sum_{j=0}^{n} w_{nj}K(t,t_j,y(t_j)) = O(h^r)$$

as $h \to 0$, $n \to \infty$, with $t_n = t_0 + nh$ fixed. An important class of quadrature formulas, which includes the well-known Gregory formulas, are the so-called (ρ,σ) - reducible quadrature formulas [14].

Our general *VLM method* for (1.1) consists of

(i) the VLM *formula*

$$(2.1.4)\qquad \theta\sum_{i=0}^{k}\alpha_i y_{n-i} + \sum_{i=0}^{k}\sum_{j=-k}^{k}\beta_{ij}Y_{n-i}(t_{n+j}) = h\sum_{i=0}^{k}\sum_{j=-k}^{k}\gamma_{ij}K_{n-i}(t_{n+j}),\quad n = k^*(1)N,$$

(k^* fixed) where α_i, β_{ij} and γ_{ij}, $i = 0(1)k$, $j = -k(1)k$, are to be prescribed, and

(ii) the *quadrature formula* (2.1.2) for the computation of $Y_{n-i}(t_{n+j})$.

In the VLM method the quantities $y_1,\ldots,y_{k^*-1}$ with $k^* = k + n_1$ are assumed to be precomputed by some starting method. Then $y_{k^*},\ldots,y_N$ can be succesively computed using (2.1.4). Since the kernel $K(t,\tau,y)$ is not necessarily defined outside S, we require $\beta_{ij} = \gamma_{ij} = 0$ for $j < -i$. Furthermore, if β_{ij}, $\gamma_{ij} \neq 0$ for $j = 1(1)k$ we assume that the domain of definition of K can be extended to points (t,τ) with $t \leq T + kh$, $\tau \leq T$. It is convenient to characterize (2.1.4) by the matrices

$$(2.1.5)\qquad A = (\alpha_i),\quad B = (\beta_{ij}),\quad C = (\gamma_{ij})$$

where the row index i assumes the values 0(1)k and the column index j the values -k(1)k. (Note that for $\theta = 0$ the values of the coefficients α_i in (2.1.4) are irrelevant.)

We now describe four subclasses of (2.1.4) used as illustrating examples in this paper.

<u>Subclass 1</u> *Direct quadrature methods*

Direct quadrature methods for (1.1) are characterized by the (1×1) matrices $A = I$, $B = -I$, $C = 0$, for which (2.1.4) reduces to the simple scheme $\theta y_n = Y_n(t_n)$, $n = n_1(1)N$.

<u>Subclass 2</u> *Indirect linear multistep methods*

In [6] methods for (1.1) (with $\theta=1$) were considered in which the VLM formula (2.1.4) is generated by the matrices

$$(2.1.6)\qquad A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix}, \quad B = \begin{bmatrix} \bigcirc & b_0\delta_0 & b_0\delta_1 & \cdots & b_0\delta_k \\ & b_1\delta_0 & b_1\delta_1 & \cdots & b_1\delta_k \\ & \vdots & \vdots & & \vdots \\ & & & & \bigcirc \\ b_k\delta_0 & & b_k\delta_1 & \cdots & b_k\delta_k \end{bmatrix},$$

$$C = \left[\begin{array}{ccc|c} \bigcirc & & b_0 & \\ & b_1 & & \\ & \cdot^{\cdot^{\cdot}} & & \bigcirc \\ b_k & & \bigcirc & \end{array}\right],$$

where the a_i and b_i, $i = 0(1)k$, are the coefficients of some LM method for ODEs and the δ_i, $i = 0(1)k$, are the coefficients of (k+1)-point forward differentiation formula (Table 1 of the Appendix lists these for $k = 1(1)5$). This VLM formula forms, together with (2.1.2), a so-called *indirect linear multistep* (ILM) method for (1.1), not only for $\theta = 1$, but also for $\theta = 0$. When the a_i and b_i are the coefficients of a backward differentiation method (for $k=1(1)5$ these are listed in Table 2 of the Appendix), (2.1.4) represents the so-called IBD (indirect backward differentiation) method, analysed in [5] (in the case $\theta=1$). For this method, we have $b_0 = 1$, $b_{<0} = 0$ and $b_0\delta_j = a_j$, $j = 0(1)k$.

It should be remarked that the ILM methods require the extension of the domain of definition S with the points $\{(t,\tau) \mid T < t \le T + kh,\ t_0 \le \tau \le T\}$.

In this connection we observe that if S can also be extended to points with $t < \tau$ we may use *backward* instead of *forward* differentiation coefficients δ_i in the IBD method, i.e., the matrix B is replaced by the matrix (β_{ij}) all elements of which vanish, except for those in the first row, which are given by $(b_0\delta_k, b_0\delta_{k-1}, \ldots, b_0\delta_0, 0, \ldots, 0)$.

Subclass 3 *Multilag methods*

In [16,17] we find methods for (1.1) with $\theta = 1$ which can be characterized by the matrices

$$(2.1.7) \qquad A = \begin{bmatrix} a_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} & 0 & \\ & a_1 & \\ \bigcirc & \vdots & \bigcirc \\ & \cdot & \\ & a_k & \end{bmatrix}, \quad C = \begin{bmatrix} & b_0 & \\ & b_1 & \\ \bigcirc & \vdots & \bigcirc \\ & \cdot & \\ & b_k & \end{bmatrix}.$$

These methods were called *multilag methods* (ML) for (1.1) with $\theta = 1$. Here, the a_i and b_i, $i = 0(1)k$ may be the coefficients of any LM method for ODEs. Wolkenfelt has pointed out that in the case that the lag term $Y_n(t)$ is computed by using a quadrature rule which is reducible to an LM-method for ODEs with the *same* coefficients a_i and b_i, then the resulting method is, in fact, equivalent to a DQ method based on the same quadrature rule (provided, of course, that identical starting values are used).

Subclass 4 *Modified multilag methods*

In [16] Wolkenfelt introduced a modification of the ML methods, viz., the so-called *modified multilag* (MML) methods for (1.1) characterized by the matrices

$$(2.1.8) \qquad A = \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{bmatrix}, \quad B = \begin{bmatrix} & & 0 & \\ \bigcirc & -a_1 & a_1 & \\ & \cdot & \cdot & \\ \cdot & \bigcirc & \cdot & \bigcirc \\ \cdot & & \cdot & \\ -a_k & & a_k & \end{bmatrix}, \quad C = \begin{bmatrix} & b_0 & \\ & b_1 & \\ \bigcirc & \vdots & \bigcirc \\ & \cdot & \\ & b_k & \end{bmatrix}.$$

The a_i and b_i are, again, the coefficients of any LM method for ODEs. A common choice are the *Adams-Moulton* formulas (listed in Table 3 in the Appendix, for k=1(1)5). As for ML methods, the MML method is algebraically equivalent with the DQ method if in both methods the lag term formula is based on the LM formula $\{a_i, b_i\}$ and if the starting values would be identical, where $y_n = Y_n(t_n)$ for $n_1 \le n \le n_1 + k$.

2.2. Consistency of VLM formulas for Volterra integral equations

Let $C_2^1(S)$ denote the space of continuous functions Y(t,s), differentiable with respect to s for all $(t,s) \in S$. Along the lines of the theory developed for LM formulas for ODEs, we associate with the VLM formula (2.1.4) the linear difference - differential operator $\underset{\sim}{L}_n : C_2^1(S) \to \mathbb{R}$, defined by

$$(2.2.1) \qquad \underset{\sim}{L}_n[Y] := \sum_{i=0}^{k} \{\theta\alpha_i Y(t_{n-i}, t_{n-i}) + \sum_{j=-k}^{k} [\beta_{ij} - \gamma_{ij} h \frac{\partial}{\partial s}] Y(t_{n+j}, t_{n-i})\}.$$

As is usual in the consistency analysis of numerical schemes for functional equations, we now substitute the *exact* solution y(t) of (1.1) into (2.1.4), and analyze the resulting residue. With the relations $\theta y(t) = Y(t,t)$ and $\partial Y(t,s)/\partial s = K(t,s,y(s))$, and (2.1.3) we obtain the equation

$$(2.2.2) \qquad \sum_{i=0}^{k} \Big\{ (\theta\alpha_i y(t_{n-i}) + \sum_{j=-k}^{k} \Big(\beta_{ij} [g(t_{n+j}) + h \sum_{\ell=0}^{n-i} w_{n-i,\ell} K(t_{n+j}, t_\ell, y(t_\ell))] - h\gamma_{ij} K(t_{n+j}, t_{n-i}, y(t_{n-i})) \Big) \Big\} = \underset{\sim}{L}_n[Y] - \sum_{i=0}^{k} \sum_{j=-k}^{k} \beta_{ij} E_{n-i}(h; t_{n+j}).$$

The second term in the residual originates from the quadrature formula {(2.1.2) - (2.1.3)}, and is $O(h^r)$ if $E_n(h;t)$ is $O(h^r)$. The first term originates from the VLM formula (2.1.4) and will be called the *local truncation error of the VLM formula*. In order to analyze it, we use the following

<u>Definition 2.2.1</u>. The operator (2.2.1) and the associated VLM formula (2.1.4) are said to be *consistent of order* p with equation (1.1) if for all $X \in C^{p+1}(S)$, which in the case of first kind equations (θ=0) *vanish on the line* $t = \tau$, we have $\underset{\sim}{L}_n[X] = O(h^{p+1})$ as $h \to 0$ with nonvanishing error constant. □

The following two theorems express the p-th order consistency conditions in terms of the parameters occurring in (2.1.4) for (1.1) in the cases $\theta = 1$ and $\theta = 0$, respectively.

THEOREM 2.2.1. *The operator* $\underset{\sim}{L}_n$ *and the associated VLM formula (2.1.4) are consistent of order* p *with* $\{(1.1),\ \theta=1\}$ *if* $C_{q\ell} = 0$ *for* $q = 0(1)p$, $\ell = 0(1)q$, *where*

$$(2.2.3)\qquad C_{q\ell} := \frac{1}{(q-\ell)!\,\ell!}\sum_{i=0}^{k}\Big[(-i)^q\alpha_i - \sum_{j=-k}^{k} j^{q-\ell}(-i)^{\ell-1}(i\beta_{ij}+\ell\gamma_{ij})\Big],$$

with the convention that $(-i)^{\ell-1}\ell = 0$ *if* $i = \ell = 0$. □

THEOREM 2.2.2. *The operator* $\underset{\sim}{L}_n$ *and the associated VLM formula (2.1.4) are consistent of order* p *with* $\{(1.1),\ \theta=0\}$ *if* $B_{q\ell} = 0$ *for* $q = 1(1)p$, $\ell = 1(1)q$, *where*

$$(2.2.4)\qquad B_{q\ell} := \frac{1}{(q-\ell)!\,\ell!}\sum_{i=0}^{k}\sum_{j=-k}^{k}(j-i)^{q-\ell-1}(j+i)^{\ell-1}\big[\beta_{ij}(j^2-i^2) - \gamma_{ij}(qj+qi-2\ell j)\big].\qquad □$$

The various orders of consistency of the illustrating subclasses introduced in Section 2.1 can now be found by substituting the relevant values of the parameters α_i, β_{ij} and γ_{ij} into the above two theorems. The results are summarized in the following corollary.

Corollary 2.2.1. *Let* $\tilde{p}$ *be the order of consistency of the LM formula for ODEs defining the coefficients* $\{a_i, b_i\}$ *in Subclasses 2, 3 and 4. Then the order of consistency* p *of the operator* $\underset{\sim}{L}_n$ *and of the associated VLM formula (2.1.4) with equation (1.1) is given by*

$$p = \begin{cases} \infty \text{ for Subclass 1 (DQ);} \\ \min\{k, \tilde{p}/\theta\} \text{ for Subclass 2 (ILM);} \\ \tilde{p} \text{ both for Subclasses 3 and 4 (ML resp. MML).} \end{cases} \qquad □$$

Note that for $\theta = 0$ in the ILM case this corollary gives $p = k$, *independent* of the order of consistency $\tilde{p}$, and, in fact, in this case the $\{a_i, b_i\}$ need not represent an LM method for ODEs at all.

Let us assume in the rest of this section that the VLM formula (2.1.4) is consistent of order p with equation (1.1). From the proofs of Theorems 2.2.1 resp. 2.2.2 it follows that the local truncation errror $\underset{\sim}{L}_n[Y]$ can be expressed in terms of the constants defined in (2.2.3) resp. (2.2.4) as follows:

(2.2.5) $$\underset{\sim}{L}_n[Y] = h^{p+1}\sum_{\ell=0}^{p+1} C_{p+1,\ell}Y^{(p+1-\ell,\ell)} + O(h^{p+2}), \text{ as } h \to 0,$$

resp.

(2.2.6) $$\underset{\sim}{L}_n[Y] = h^{p+1}\sum_{\ell=1}^{p+1} B_{p+1,\ell}Z^{(p+1-\ell,\ell)} + O(h^{p+2}), \text{ as } h \to 0,$$

where

$$Y^{(i,j)} := (\partial/\partial t)^i(\partial/\partial s)^j Y(t,s)|_{t=s=t_n},$$

$$Z^{(i,j)} := (\partial/\partial u)^i(\partial/\partial v)^j Z(u,v)|_{u=2t_n, v=0},$$

$$Z(u,v) := Y(\tfrac{u+v}{2},\tfrac{u-v}{2}).$$

In order to compare the values of the error constants $C_{p+1,\ell}$ and $B_{p+1,\ell}$ for the various subclasses introduced in Section 2.1 we have evaluated and simplified the expressions for these constants as much as possible, and obtained the following results.

For $\theta = 1$ Corollary 2.2.1 gives for the ILM formula: $p = k$ provided that $\tilde{p} \geq k$ (which is a reasonable assumption, true, e.g., when the LM formula for ODEs is a Backward Differentiation formula ($\tilde{p}=k$) or an Adams-Moulton formula ($\tilde{p}=k+1$)). Hence,

$$C_{p+1,\ell} = \frac{(-1)^{p-1}}{(p+1-\ell)!\,\ell!}\sum_{i=0}^{k}\{i^p[ia_i+(p+1)b_i]-R\}, \quad p = k,$$

with $R = k!b_i$ if $\ell = 0$ and $R = 0$ if $\ell = 1,2,\ldots,p+1$. For the (M)ML formula, Corollary 2.2.1 gives $p = \tilde{p}$ and we found

$$C_{p+1,\ell} = \begin{cases} 0 & \text{if } \ell \le p, \\ \dfrac{(-1)^{p-1}}{(p+1-\ell)!\ell!} \sum_{i=0}^{k} i^p[ia_i+(p+1)b_i] & \text{if } \ell = p+1, \end{cases} \quad p = \tilde{p}.$$

In Table 2.2.1 the numerical values of the constants $C_{p+1,\ell}$ are explicitly given for two popular choices of the coefficients $\{a_i,b_i\}$, viz., the BD formulas and the AM formulas.

Table 2.2.1. Error constants $C_{p+1,\ell} = C^*_{p+1,\ell}\,\dfrac{1}{\ell!(p+1-\ell)!}$, $\ell = 0(1)p+1$

VLM[LM]	p		k=1	k=2	k=3	k=4	k=5
ILM[BD]	k	$C^*_{p+1,0}$	$= -2$	0	$-\frac{72}{11}$	0	$-\frac{14400}{137}$
		$C^*_{p+1,\ell>0}$	$= -1$	$-\frac{4}{3}$	$-\frac{36}{11}$	$-\frac{288}{25}$	$-\frac{7200}{137}$
ILM[AM]	k	$C^*_{p+1,0}$	$= -1$	2	-6	24	-120
		$C^*_{p+1,\ell>0}$	$= 0$	0	0	0	0
(M)ML[BD]	k	$C^*_{p+1,\ell<p+1}$	$= 0$	0	0	0	0
		$C^*_{p+1,p+1}$	$= -1$	$-\frac{4}{3}$	$-\frac{36}{11}$	$-\frac{288}{25}$	$-\frac{7200}{137}$
(M)ML[AM]	k+1	$C^*_{p+1,\ell<p+1}$	$= 0$	0	0	0	0
		$C^*_{p+1,p+1}$	$= -\frac{1}{2}$	-1	$-\frac{19}{6}$	$-\frac{27}{2}$	$-\frac{863}{12}$

For $\theta = 0$ and in the case of the ILM formula, Corollary 2.2.1 gives $p = k$, while the coefficients $\{b_i\}$ can still be chosen freely. For the error constants we found

$$B_{p+1,\ell} = \frac{(-1)^k k!}{(p+1-\ell)!\ell!} \sum_{i=0}^{k} b_i, \quad 1 \le \ell \le p+1, \quad p = k.$$

For the MML formula, Corollary 2.2.1 gives $p = \tilde{p}$, and for the error constants we found

$$B_{p+1,\ell} = \frac{(-1)^{p+1-\ell}}{(p+1-\ell)!\ell!} \sum_{i=0}^{k} i^p[ia_i + (p+1)b_i], \qquad 1 \le \ell \le p+1.$$

In Table 2.2.2 the $B_{p+1,\ell}$ are given for the BD formulas (cf. Table 2.2.1).

Table 2.2.2 Error constants $B_{p+1,\ell} = B^*_{p+1,\ell} \frac{1}{\ell!(p+1-\ell)!}$,

$-\ \ell = 1(1)p+1$

VLM[LM]	p			k=1	k=2	k=3	k=4	k=5
ILM[BD]	k	$B^*_{p+1,\ell}$	=	-1	$\frac{4}{3}$	$-\frac{36}{11}$	$\frac{288}{25}$	$-\frac{7200}{137}$
MML[BD]	k	$(-1)^{p+1-\ell}B^*_{p+1,\ell}$	=	-1	$\frac{4}{3}$	$-\frac{36}{11}$	$\frac{288}{25}$	$-\frac{7200}{137}$

2.3. Convergence

We first give a definition of convergence.

Definition 2.3.1. A VLM method is said to yield a *convergent solution for* (1.1) if $y_n \to y(t_n)$ as $h \to 0$, with $t_n = t$ fixed, holds for all convergent starting values $\{y_i, Y_i(t_j)\}$, $i = 1,\dots,k-1$, $j = -i,-i+1,\dots,k-i$. □

Before considering the convergence of VLM methods for (1.1) we answer the question to what equation the numerical scheme (2.1.4) converges if we substitute a sufficiently differentiable function y(t) (not neccessarily the exact solution) and if we then let h tend to zero in a fixed point. To that end, we define the polynomial

$$(2.3.1)\qquad \alpha(z) := \sum_{i=0}^{k} \alpha_i z^{k-i}$$

and the quantities

$$(2.3.2)\qquad A_q := \sum_{i=0}^{k} (-i)^q \alpha_i, \quad q = 0,1,\ldots .$$

We observe that $A_0 = \alpha(1)$, $A_1 = \alpha'(1) - k\alpha(1),\ldots$.

THEOREM 2.3.1. *If sufficiently differentiable functions* $y(t)$, $g(t)$ *and* $K(t,\tau,y)$ *are substituted into the VLM method, then the method converges to the equation*

$$(2.3.3)\qquad \sum_{q=0}^{m} h^q\{\frac{\theta A_q}{q!}\frac{d^q}{dt^q}y(t) + \sum_{\ell=0}^{q}[C_{q\ell} - \frac{A_q}{\ell!(q-\ell)!}](\frac{\partial}{\partial t})^{q-\ell}(\frac{\partial}{\partial s})^{\ell}Y(t,t)\}$$

$$= O(h^r + h^{m+1}) \quad \text{as } h \to 0$$

where r *is defined in (2.1.3),* $C_{q\ell}$ *in (2.2.3), and* m *is some integer* ≥ 0 *determined by the differentiability of* y, g *and* K. □

Examples. In the case of the DQ method for $\{(1.1), \theta=1\}$ we have $A_0 = 1$, so that we infer from Corollary 2.2.1 and Theorem 2.3.1 that the numerical scheme converges to the equation $y(t) - Y(t,t) = 0$ as $h \to 0$, which is the original equation (1.1). In the case of the DQ method for $\{(1.1), \theta=0\}$ we have $C_{00} = 1$, so that it easily follows that the numerical scheme converges as $h \to 0$ to the equation $Y(t,t) = 0$, also the original equation (1.1). In the case of the ILM method for (1.1) it is not difficult to show that if the coefficients $\{a_i, b_i\}$ in (2.1.6) correspond to a convergent LM formula for ODEs, then the numerical scheme converges, as $h \to 0$, to the *differentiated* Volterra equation $\theta y'(t) = K(t,t,y(t)) + Y_t(t,t)$. □

In order to present convergence theorems for VLM methods, we need the following concepts and definitions: A polynomial is called *simple von Neumann* if its zeros lie on the unit disk, those on the unit *circle* being simple. A polynomial is called *Schur* if its zeros lie *within* the unit circle. Besides $\alpha(z)$ defined above, we define

$$\text{(2.3.4a)} \qquad \beta(z) := \sum_{i=0}^{k} \beta_i z^{k-i} \text{ where } \beta_i := \sum_{j=-i}^{k-i} \beta_{ij};$$

$$\text{(2.3.4b)} \qquad \gamma(z) := \sum_{i=0}^{k} \gamma_i z^{k-i} \text{ where } \gamma_i := \sum_{j=-i}^{k-i} \gamma_{ij}.$$

Furthermore, we need

$$\text{(2.3.5)} \qquad b := \max_{i,j} |\beta_{ij}|, \; c := \max_{i,j} |\gamma_{ij}|, \; w := \max_{i,j} |w_{ij}|;$$

$$\text{(2.3.6a)} \qquad \begin{cases} \Delta K(t,s,y,y^*) := K(t,s,y) - K(t,s,y^*), \\ \Delta E(h) := \max\limits_{\substack{i \le j \le N \\ \ell \le k}} |E_i(h;t_j) - E_i(h;t_{j+\ell})|, \end{cases}$$

$$\text{(2.3.6b)} \qquad \begin{cases} E(h) := \max\limits_{\substack{i \le j \le N \\ \ell \le k}} |E_i(h;t_{j+\ell})|, \\ T(h) := \max\limits_{i \le N} |\underset{\sim}{L}_i(Y)|, \\ \delta(h) := \max\limits_{1 \le j \le k^*-1} |y(t_j) - y_j|, \end{cases}$$

where $y(t)$ is the exact solution of (1.1) resp. (1.2) and $Y(t,s)$, $Y_n(t)$ are the corresponding functions defined in (1.5) resp. (2.1.2). $E(h)$ is the maximal error arising in the approximation of the lag terms $Y_n(t)$ during the integration process until $t = T$, $T(h)$ may be considered as the maximal *local truncation error* of the VLM formula (2.1.4) until $t = T$, and $\delta(h)$ is the maximal starting error.

2.3.1. Second kind equations

We are now in a position to state a general convergence theorem for VLM methods in the case of *second* kind equations ($\theta=1$), which provides an estimate for the *global error*

$$\text{(2.3.7)} \qquad \varepsilon_n := y(t_n) - y_n.$$

THEOREM 2.3.2. *Let the conditions for the existence of a unique solution* $y \in C(I)$ *of* $\{(1,1),\theta=1\}$ *be satisfied (see Section 1).*
(a) If $\alpha(z) \equiv \alpha_0 z^k$, $\alpha_0 \neq 0$, *then there exists a constant* $C > 0$, *independent of* h, *such that for* h *sufficiently small*

$$|\varepsilon_n| \le C[h\delta(h) + E(h) + T(h)], \quad n = k^*,\dots,N.$$

(b) If $\alpha(z)$ *is simple von Neumann, if* $\beta(z) \equiv 0$, *and if* ΔK *satisfies the (uniform) Lipschitz condition*

$$|\Delta K(t,s,y,y^*) - \Delta K(t^*,s,y,y^*)| \le L|t-t^*||y-y^*|,$$

for all (t,s,y), (t,s,y^*), (t^*,s,y), $(t^*,s,y^*) \in S\times\{|y|<\infty\}$, *where the Lipschitz constant* L *is independent of* t,t^*,y *and* y^*, *then there exists a constant* $C > 0$, *independent of* h, *such that for* h *sufficiently small*

$$|\varepsilon_n| \le Ch^{-1}[h\delta(h) + \Delta E(h) + T(h)], \quad n = k^*,\dots,N. \quad \square$$

Using Theorem 2.3.2 it is easy to derive the orders of convergence of the subclasses introduced in Section 2.1. The results are given in the following

Corollary 2.3.1. *Let* $\tilde{p}$ *be the order of consistency of the LM formula for ODEs defining the coefficients* $\{a_i,b_i\}$ *employed in the ILM and (M)ML methods, let* $\delta(h) = O(h^s)$, $E(h) = O(h^r)$ *and* $\Delta E(h) = O(h^{r+1})$ *as* $h \to 0$. *Then the order of convergence* p *is given by*

$$p = \begin{cases} \min\{s+1,r\} & \text{for the DQ method} \\ \min\{s,r,\tilde{p},k\} & \text{for the ILM method} \\ \min\{s+1,r,\tilde{p}+1\} & \text{for the ML method} \\ \min\{s,r,\tilde{p}\} & \text{for the MML method} \end{cases} . \quad \square$$

The convergence analysis of the DQ methods goes back to KOBAYASI [7], LINZ [9] and NOBLE [13]. The (M)ML methods were proved to be convergent in WOLKENFELT [16].

It is known (cf. [6,16]) that VLM methods which have $\beta(z) \equiv 0$ behave more stable then DQ methods if large Lipschitz constants for $\partial K/\partial y$ are involved. The *maximal* attainable order of convergence of these VLM methods is expressed in the following

Corollary 2.3.2. *Let* $\delta(h) = O(h^s)$, $\Delta E(h) = O(h^{r+1})$ *as* $h \to 0$, *let* $\alpha(z)$ *be simple von Neumann and let* $\beta(z) \equiv 0$. *Then the order of convergence* p *of the* k*-step VLM method* {(2.1.4); (2.1.2)} *satisfies*

$$p \le \begin{cases} \min(s,r,k+1) & \text{for } k \text{ odd,} \\ \min(s,r,k+2) & \text{for } k \text{ even.} \end{cases} \quad \square$$

From this corollary it follows that the MML methods are of maximal attainable order of convergence if we choose the generating LM formula $\{\rho,\sigma\}$ to be optimal, that is, of order $k + 1$ when k is odd and of order $k + 2$ when k is even. We note that the restriction $p \le k$ in the ILM methods is due to the use of a k-step forward differentiation formula $\{\delta_i\}$ in the generating matrix B (see (2.1.6)).

2.3.2 First kind equations

Now we shall give convergence theorems for VLM methods for Volterra *first* kind equations ($\theta=0$). We restrict our attention here to *linear* equations, i.e., we assume in (1.1) that

$$K(t,\tau,y(\tau)) = K(t,\tau)y(\tau). \tag{2.3.8}$$

We first give the following convergence theorem of WOLKENFELT [14] for (ρ,σ)-reducible DQ methods.

THEOREM 2.3.3. *Let the conditions for the existence of a unique solution* $y(t) \in C(I)$ *of* $\{(1.1), \theta=0\}$ *be satisfied (see Section 1), where* $K(t,\tau,y)$ *is of the form given in (2.3.8). Let* $A = 0$, $B = 1$, $C = 0$ *in (2.1.5) (DQ formula) and let the weights in (2.1.2) be given by a* (ρ,σ)*-reducible quadrature formula of order* $r \ge 1$, *where* σ *is simple von Neumann. Then there exists a constant* $C > 0$, *independent of* h, *such that for* h *sufficiently* *small*

$$|\varepsilon_n| \le C[\delta(h) + E(h)], \quad n = k,\dots,N. \quad \square$$

PROOF. See [14].

THEOREM 2.3.4. *Let the conditions for the existence of a unique solution* $y \in C(I)$ *of* $\{(1.1), \theta=0\}$ *be satisfied . Let* $\beta(z) \equiv 0$ *and let* $\gamma(z)$ *be Schur. Then there exists a constant* $C > 0$, *independent of* h *such that for* h *sufficiently small*

$$|\varepsilon_n| \le Ch^{-1}[h\delta(h) + \Delta E(h) + T(h)], \; n = k^*,\dots,N. \quad \square$$

Observe that this convergence result is identical to that obtained for VLM methods for $\{(1.1), \theta=1\}$ with $\beta(z) \equiv 0$ (Theorem 2.3.2(b)). Now it is easy to derive from Theorems 2.3.3 and 2.3.4 the orders of convergence of the DQ, ILM and MML methods for $\{(1.1), \theta=0\}$. The results are summarized in the following

Corollary 2.3.3. *Let* $\bar{p}$ *be the order of consistency of the LM formula* $\{\rho,\sigma\}$ *employed in the DQ lag term formula (2.1.2) and let* $\tilde{p}$ *be the order of consistency of the LM formula for ODEs employed in the MML formula (2.1.8) (with* $A\equiv 0$*). Furthermore, let* $\delta(h) = O(h^s)$, $E(h) = O(h^r)$ *and* $\Delta E(h) = O(h^{r+1})$ *as* $h \to 0$. *Then the order of convergence* p *is given by*

$$p = \begin{cases} \min\{s,\bar{p}\} & \text{for the DQ method with } \sigma \text{ being simple von Neumann,} \\ \min\{s,r,k\} & \text{for the ILM method with } \gamma \text{ being Schur,} \\ \min\{s,r,\tilde{p}\} & \text{for the MML method with } \gamma \text{ being Schur.} \end{cases} \quad \square$$

WOLKENFELT [15] has also given a convergence theorem for MML methods for *nonlinear* equations $\{(1.1), \theta=0\}$, with the following restrictions on the coefficients β_{ij} and γ_{ij}: $\beta(z) \equiv 0$, $\gamma_{00} \neq 0$ and all other γ_{ij} vanish.

2.4 Numerical experiments

In this section we present the results of numerical experiments in order to support and illustrate the convergence behaviour of VLM methods for (1.1) as predicted by Corollaries 2.3.1 and 2.3.3, by applying various DQ, ILM and (M)ML methods to a number of problems.

The required starting values for y_i, $0 \le i < n_1 + k$, are taken from the exact solution $y(t_i)$ (so that $s=\infty$ in Corollaries 2.3.1 and 2.3.3), and values of the lag term $Y_n(t)$ required in (2.1.4) for $n \ge n_1$ are computed with a Gregory quadrature rule in (2.1.2) of the proper order. The coefficients $\{a_i, b_i\}$ in the ILM and (M)ML formulas are taken from Tables 2 and 3 of the Appendix. The values of r, $\tilde{p}$ and k in Corollary 2.3.1 and of r, $\bar{p}$, $\tilde{p}$ and k in Corollary 2.3.3 are chosen as small as is allowed by the theoretical order to be tested.

In the tables of results we present the number of correct significant digits in the end point T, i.e., the value of

$$(2.1.4) \qquad sd(h) := -\log_{10}(|y(T)-y_N|/|y(T)|), \quad T = t_N = Nh.$$

Moreover, we list the *effective* order of the method, viz., the value of $(sd(h)-sd(2h))/\log_{10}2$. This value should tend to the asymptotic order of convergence as $h \to 0$ and will tell us therefore

(i) whether the asymptotic, theoretical order of the numerical scheme is correct, and

(ii) how fast the asymptotic order is reached.

2.1.4. Second kind equations ((1.1) with $\theta=1$)

Example 2.1.4 (GAREY [2], adapted)

$$K(t,\tau,y) = -\lambda \cdot \ln(1+t-\tau)y,$$

$$g(t) = 1 - t + \lambda[\tfrac{1}{2}(1-t^2)\ln(1+t)+\tfrac{3}{4}t^2 - \tfrac{1}{2}t],$$

$$y(t) = 1 - t,$$

$$[t_0,T] = [0,4].$$

For $\lambda = 4$ and $\lambda = 100$ Tables 2.4.1 and 2.4.2 give the results obtained with DQ, ILM and (M)ML methods of asymptotic order 5, where for the coefficients $\{a_i, b_i\}$ employed in the ILM and (M)ML methods we used the coefficients of the Adams-Moulton formula of the proper order. G_r means that for the lag term we used a Gregory rule of order r and $AM_{\tilde{p}}$ means that a $\tilde{p}$-th order Adams-Moulton formula was used.

Table 2.4.1 Example 2.4.1 with $\lambda = 4$

h	$DQ(G_5)$	$ILM(G_5\text{-}AM_6)$	$ML(G_5\text{-}AM_4)$	$MML(G_5\text{-}AM_5)$
1/4	4.6	3.4	4.3	6.1
	>4.6	> 3.8	>4.8	>3.9
1/8	6.0	4.5	5.7	7.3
	> 5.0	> 4.5	>4.6	>3.2
1/16	7.5	5.9	7.1	8.2
	> 5.0	> 4.7	>5.0	>4.0
1/32	9.0	7.3	8.6	9.4
	> 5.0	> 4.8	>5.0	>4.6
1/64	10.5	8.8	10.1	10.8

Table 2.4.2 Example 2.4.1 with $\lambda = 100$

h	$DQ(G_5)$	$ILM(G_5\text{-}AM_6)$	$ML(G_5\text{-}AM_4)$	$MML(G_5\text{-}AM_5)$
1/4	-6.5	1.8	-3.7	-2.4
	>30	>8.9	>25	>22
1/8	2.3	4.5	3.7	4.2
	>13	>4.1	>8.2	>16
1/16	6.3	5.8	6.2	9.0
	>5.8	>4.4	>4.7	>2.3
1/32	8.1	7.1	7.6	9.7
	>6.8	>6.4	>5.8	>2.5
1/64	10.1	9.0	9.3	10.4

The example with $\lambda = 4$ shows that the correct asymptotic order $p = 5$ is reached by all methods for not too small integration steps. For relatively large values of h the MML method shows the most accurate results. The ILM method shows an accuracy about 1 - 2 digits less than the other methods, due to larger error constants (cf. Table 2.2.1). The example with $\lambda = 100$ shows that the ILM method is stable for "realistic" values of h (in view of the behaviour of the exact solution, integration steps $h = 1/4$ or $h = 1/8$ should be small enough for representing the function $y(t) = 1-t$), whereas the other methods develop instabilities. If h is decreased the (M)ML methods become more accurate than the ILM method. Except for the ML method this experiment does not yet demonstrate the asymptotic order $p = 5$.

Thus, we conclude that the (M)ML methods are superior for *nonstiff* problems ($h|\partial K/\partial y|$ small), and the ILM methods superior for stiff problems ($h|\partial K/\partial y|$ large).

2.4.2 First kind equations ((1.1) with $\theta=0$)

Example 2.4.2 (GLADWIN [3])

$$K(t,\tau,y) = \cos(t-\tau)y,$$
$$g(t) = -\exp(t) - \sin(t) + \cos(t),$$
$$y(t) = \exp(t),$$
$$[t_0,T] = [0,4].$$

Table 2.4.3 gives the results obtained with DQ, ILM and MML methods of asymptotic order 4 and 5, where for the coefficients $\{a_i,b_i\}$ in the ILM and MML methods we used the coefficients of the backward differentiation formulas of the proper order. BD_k means that a k-step (k-th order) BD formula was used.

Table 2.4.3 Example 2.4.2

h	$DQ(G_4)$	$ILM(G_4-BD_4)$	$MML(G_4-BD_4)$	$DQ(G_5)$	$ILM(G_5-BD_5)$	$MML(G_5-BD_5)$
1/10	-7.6	4.3	3.9	-11	5.6	4.9
		>4.1	>3.8		>4.9	>4.9
1/20	-21	5.5	5.1	-29	7.0	6.4
		>4.0	>3.9		>5.0	>5.0
1/40	-50	6.7	6.3	-65	8.5	7.9
		>4.0	>4.0		>5.2	>5.0
1/80	-109	7.9	7.5	-140	10.1	9.4

For the ILM and the MML methods the results show that the correct asymptotic order is reached already for relatively large values of h. The apparent unstable behaviour of the $DQ(G_4)$ and $DQ(G_5)$ methods is explained by the fact that the Gregory quadrature formulas of order ≥ 3 are (ρ,σ)-quadrature formulas for which the σ-polynomial is *not* simple von Neumann (cf. Corollary 2.3.3). Unlike its performance for second kind equations, the ILM method is here more accurate than the MML method.

3. VLM METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

3.1. The general VLM method

In analogy to the VLM formula (2.1.4) for Volterra integral equations we formally define the VLM formula for Volterra integro-differential equations (1.2) as follows:

$$\text{(3.1.1a)} \qquad \sum_{i=0}^{k} \alpha_i^* y_{n-i} = h \sum_{i=0}^{k} \gamma_i^* f_{n-i}, \quad f_n := f(t_n, y_n, z_n),$$

$$\text{(3.1.1b)} \qquad \sum_{i=0}^{k} \alpha_i z_{n-i} + \sum_{i=0}^{k} \sum_{j=-k}^{k} \beta_{ij} Y_{n-i}(t_{n+j}) = h \sum_{i=0}^{k} \sum_{j=-k}^{k} \gamma_{ij} K_{n-i}(t_{n+j}),$$

$$n = k^*(1)N,$$

where $Y_n(t)$ is defined as in (2.1.2) and $\{\alpha_i^*, \gamma_i^*\}_{i=0}^{k}$ are the coefficients of some LM method for ODEs. These formulas (3.1.1), combined with (2.1.2), will be called a *VLM method* for integro-differential equations. Formula

(3.1.1b) can be characterized by the parameter matrices $A = (\alpha_i)$, $B = (\beta_{ij})$, $C = (\gamma_{ij})$.

When we compare (3.1.1) with (2.1.4), it is clear that *all* methods defined for second kind Volterra integral equations by specifying the matrices A, B and C in (2.1.5) and the quadrature weights w_{nj} in (2.1.2), can be extended to methods for Volterra integro-differential equations by specifying the coefficients $\{\alpha_i^*, \beta_i^*\}$ of some LM method for ODEs. In this way, we define DQ, ILM, ML and MML methods for (1.2) where the matrices A, B and C are specified in Section 3.1, Subclasses 1, 2, 3 and 4, respectively. For example, any DQ method for (1.2) is specified by $A = 1$, $B = -1$, $C = 0$, which gives $z_n = Y_n(t_n)$ for (3.1.1b).

An alternative way to arrive at the VLM formulas (3.1.1) is obtained as follows. We first integrate (1.2) formally, which results in the *system* of Volterra integral equations of the second kind

$$(3.1.2) \qquad \begin{cases} y(t) = y(t_0) + \int_{t_0}^{t} f(\tau, y(\tau), z(\tau))d\tau, \\ z(t) = g(t) + \int_{t_0}^{t} K(t, \tau, y(\tau))d\tau. \end{cases}$$

Next, we apply the VLM formula (2.1.4) to this system with parameter matrices (A^*, B^*, C^*) and (A,B,C) for the respective components, i.e.,

$$(3.1.1') \qquad \begin{aligned} &\sum_{i=0}^{k} \alpha_i^* y_{n-i} + \sum_{i=0}^{k} \sum_{j=-k}^{k} \beta_{ij}^* Y_{n-i}^*(t_{n+j}) = h \sum_{i=0}^{k} \sum_{j=-k}^{k} \gamma_{ij}^* f(t_{n-i}, y_{n-i}, z_{n-i}) \\ &\sum_{i=0}^{k} \alpha_i z_{n-i} + \sum_{i=0}^{k} \sum_{j=-k}^{k} \beta_{ij} Y_{n-i}(t_{n+j}) = h \sum_{i=0}^{k} \sum_{j=-k}^{k} \gamma_{ij} K(t_{n+j}, t_{n-i}, y_{n-i}), \end{aligned}$$

where $Y_n^*(t_j)$ is an approximation to

$$Y^*(t,s) := y(t_0) + \int_{t_0}^{s} f(\tau, y(\tau), z(\tau))d\tau$$

at $t = t_j$, $s = t_n$. Here, however, both Y^* and f do *not* depend on t so that, by putting $\beta_i^* := \Sigma_{j=-k}^{k} \beta_{ij}^* = 0$ and writing $\gamma_i^* := \Sigma_{j=-k}^{k} \gamma_{ij}^*$ we have reduced (3.1.1') to (3.1.1).

3.2. Consistency of VLM formulas for integro-differential equations

With the numerical schemes (3.1.1) we associate the linear difference-differential operators $\underset{\sim}{L}_n^* : C^1(I) \to \mathbb{R}$ and $\underset{\sim}{L}_n : C_2^1(S) \to \mathbb{R}$, defined by

(3.2.1a) $$\underset{\sim}{L}_n^*[y] := \sum_{i=0}^{k} [\alpha_i^* - \gamma_i^* h \frac{d}{dt}]\, y(t_{n-i})$$

and

(3.2.1b) $$\underset{\sim}{L}_n[Y] := \sum_{i=0}^{k} \left\{\alpha_i Y(t_{n-i}, t_{n-i}) + \sum_{j=-k}^{k} [\beta_{ij} - \gamma_{ij} h \frac{\partial}{\partial s}]\, Y(t_{n+j}, t_{n-i})\right\},$$

where y and Y are arbitrary functions from $C^1(I)$ and $C_2^1(S)$, respectively. Now we substitute the exact solution y(t) and z(t) of (1.2) into (3.2.1) and obtain (cf. (2.2.2))

$$\sum_{i=0}^{k} [\alpha_i^* y(t_{n-i}) - \gamma_i^* h f(t_{n-i}, y(t_{n-i}), z(t_{n-i}))] = \underset{\sim}{L}_n^*[y],$$

$$\sum_{i=0}^{k} \left\{\alpha_i z(t_{n-i}) + \sum_{j=-k}^{k} [\beta_{ij} \tilde{Y}_{n-i}(t_{n+j}) - \gamma_{ij} h \tilde{K}_{n-i}(t_{n+j})]\right\} =$$

$$= \underset{\sim}{L}_n[Y] - \sum_{i=0}^{k} \sum_{j=-k}^{k} \beta_{ij} E_{n-i}(h; t_{n+j})$$

$$= \underset{\sim}{L}_n[Y] + O(h^r) \text{ as } h \to 0,$$

where $\tilde{Y}_{n-i}(t_{n+j})$ and $\tilde{K}_{n-i}(t_{n+j})$ are defined by (2.1.1) and (2.1.2) with y_n replaced by $y(t_n)$ and when r is the order of the quadrature error E_n. This shows the connection of the operators (3.2.1) with the VLM formula (3.1.1). The quantities $\underset{\sim}{L}_n^*[y]$ and $\underset{\sim}{L}_n[Y]$, with y and Y corresponding to the exact solution of (1.3), are called the *local truncation errors* of the VLM formulas (3.1.1). In analogy to Section 2.2 we use the following

Definition 3.2.1. The operators (3.2.1) and the associated VLM formulas (3.1.1) are said to be consistent of order p^* and p with the equations (1.2) if for all $y \in C^{p^*+1}(I)$ and for all $Y \in C^{p+1}(S)$, we have $\underset{\sim}{L}^*[y] = O(h^{p^*+1})$ and $L_n[Y] = O(h^{p+1})$ as $h \to 0$, with non-vanishing error constants. □

Since $\underset{\sim}{L}_n^*$ is of the same form as the linear operator $\underset{\sim}{L}_n$ occurring in ODE theory (compare LAMBERT [8, p. 23]) the consistency conditions for $\underset{\sim}{L}_n^*$ are also of the same form. (It should be remarked that in the derivation of the consistency conditions we expand $y(t_{n-i})$ and $y'(t_{n-i})$ as Taylor series about t_n, whereas Lambert expands about t_{n-k}.) Similarly, since $\underset{\sim}{L}_n$ defined in (3.2.1b) is identical to the operator defined in (2.2.1), the consistency conditions for $\underset{\sim}{L}_n$ are also known already. Therefore, the following consistency theorem is immediate.

THEOREM 3.2.1. *The operators* $\underset{\sim}{L}_n^*$ *and* $\underset{\sim}{L}_n$ *and the associated VLM formulas (3.1.1) are consistent of order* p^* *and* p *with (1.2) if* $C_q = 0$ *for* $q = 0,1,\ldots,p^*$ *and* $C_{q\ell} = 0$ *for* $q = 0,1,\ldots,p$, $\ell = 0,1,\ldots,q$, *where*

$$C_q := \frac{(-1)^q}{q!} \sum_{i=0}^{k} i^{q-1}[i\alpha_i^* + q\gamma_i^*]$$

and where $C_{q\ell}$ *is defined in (2.2.3).* □

Evidently, p^* equals the order of consistency of the LM method for ODEs with coefficients $\{\alpha_i^*, \gamma_i^*\}$. Furthermore, in the case of the DQ, ILM, ML and MML formulas, p is determined by the expressions as derived for the operator $\underset{\sim}{L}_n$ for second kind Volterra integral equations in Corollary 2.2.1.

The values of the error constants C_{p^*+1} and $C_{p+1,\ell}$, $0 \le \ell \le p+1$, follow easily from those given in Table 2.2.1 (for a number of popular methods for second kind Volterra integral equations).

3.3. Convergence

As we did for the Volterra equations of first and second kind we first consider the continuous problem to which the VLM method {(3.1.1); (2.1.2)} converges as $h \to 0$. We assume that the LM formulas in (3.1.1) are consistent and that $A_1 = \alpha'(1) - k\alpha(1) \neq 0$ (see Section 2.3). Then, for sufficiently smooth functions g, K and f the VLM method converges to the equations

$$(3.3.1)\qquad \begin{aligned} &y'(t) = f(t,y(t),z(t)) \\ &\alpha(1)[z(t) - Y(t,t)] + (\alpha'(1)-k\alpha(1))h[z'(t) - Y_t(t,t) - Y_s(t,t)] = 0 \end{aligned}$$

as $h \to 0$ (see the proof of Theorem 2.3.1). Thus, if $\alpha(1) \neq 0$ (DQ and ML method), then the VLM method is a direct discretization of (1.3). If $\alpha(1) = 0$ (ILM and MML) (and $\alpha'(1) \neq 0$ by assumption), then the linear method converges to the system

$$
\begin{aligned}
&y'(t) = f(t,y(t),z(t)) \\
(3.3.2)\quad & \\
&z'(t) = K(t,t,y(t)) + g'(t) + \int_{t_0}^{t} K_t(t,\tau,y(\tau))d\tau,
\end{aligned}
$$

that is the system (1.3) where the expression for $z(t)$ is differentiated with respect to t.

Next we present a general convergence theorem. In the proof it is convenient to use, in addition to the notation introduced in Section 2.3, the notations

$$
\begin{aligned}
&\eta_n = z(t_n) - z_n, \\
&\Delta f_n := f(t_n,y(t_n),z(t_n)) - f(t_n,y_n,z_n), \\
(3.3.3)\quad &T^*(h) := \max_{i \le N} |\underset{\sim}{L}^*_i[y]|, \quad \delta^*(h) := \max_{j \le k-1} |z(t_j) - z_j|, \\
&\alpha^*(z) := \Sigma_{i=0}^{k} \alpha_i^* z^{k-i},
\end{aligned}
$$

where z and y correspond to the exact solution.

<u>THEOREM 3.3.1</u>. *Let the conditions for the existence of a unique solution* $y \in C^1(I)$ *of (1.2) be satisfied (see Section 1). Let* $\alpha(z)$ *and* $\alpha^*(z)$ *be simple von Neumann.*

(a) If $\alpha(z) = \alpha_0 z^k$, $\alpha_0 \neq 0$, *then there exists a constant* $C > 0$, *independent of* h *such that for* h *sufficiently small*

$$|\varepsilon_n| \le C[\delta(h) + h\delta^*(h) + E(h) + T(h) + h^{-1}T^*(h)], \quad n = k^*,\dots,N.$$

(b) If $\beta(z) \equiv 0$, *then there exists a constant* $C > 0$, *independent of* h *such that for* h *sufficiently small*

$$|\varepsilon_n| \le C[\delta(h) + \delta^*(h) + h^{-1}\Delta E(h) + h^{-1}T(h) + h^{-1}T^*(h)], \quad n = k^*,\dots,N. \qquad \square$$

Using this theorem, it is easy to derive the orders of convergence of the various examples of VLM methods for Volterra integro-differential equations described in Section 3.1. The results are given in the following

Corollary 3.3.1. *Let* p^* *and* $\tilde{p}$ *be the order of consistency of the VLM formula* $\{\alpha_i^*,\gamma_i^*\}$ *employed in (3.1.1a) and of the LM formula* $\{a_i,b_i\}$ *employed in the (M)ML and ILM methods, respectively; let* $\delta(h) = O(h^s)$, $\delta^*(h) = O(h^{s^*})$, $E(h) = O(h^r)$, $\Delta E(h) = O(h^{r+1})$ *as* $h \to 0$. *Then the order of convergence* p *of the VLM method* {(3.1.1); (2.1.2)} *is given by*

$$p = \begin{cases} \min(s,s^*+1,r,p^*) & \text{for the DQ method} \\ \min(s,s^*+1,r,p^*,\tilde{p}+1) & \text{for the ML method} \\ \min(s,s^*,r,p^*,\tilde{p}) & \text{for the MML method} \\ \min(s,s^*,r,p^*,\tilde{p},k) & \text{for the ILM method.} \end{cases} \quad \Box$$

The convergence of the conventional DQ method has already been studied by LINZ [10] and MOCARSKY [12]. The convergence results for the (M)ML methods has already been given in WOLKENFELT [16].

3.4. Numerical experiments

In order to illustrate the convergence behaviour of VLM methods for (1.2) we have tested the DQ, ILM and (M)ML methods of orders 2, 3 and 4. For the two ODE-LM formulas involved in (3.1.1) we chose the backward differentiation formulas. As in the experiments for (1.1), the lag term $Y_n(t)$ was evaluated with a Gregory quadrature rule of the proper order.

Example 3.4.1. (LINZ [10], MOCARSKY [12], MAKROGLOU [11])

$$f(t,y,z) = 1 - t\cdot\exp(-t^2) + y - 2z, \quad y(0) = 0,$$

$$K(t,\tau,y) = t\tau\exp(-y^2), \quad g(t) = 0,$$

$$y(t) = t, \quad [t_0,T] = [0,2]$$

Table 3.4.1 gives the results of our experiments. The ILM method is the less accurate one, the DQ and (M)ML methods exhibit a varying accuracy behaviour.

Table 3.4.1 Example 3.4.1

second order methods (with $\{\alpha_i^*, \gamma_i^*\}=BD_2$ in (3.1.1a))

h	$DQ(G_2)$	$ILM(G_2-BD_2)$	$ML(G_2-BD_1)$	$MML(G_2-BD_2)$
1/10	2.2	3.3	2.2	1.8
	>2.0	>-2	>2.2	>2.0
1/20	2.8	2.6	2.8	2.4
	>2.0	>1.3	>2.0	>2.0
1/40	3.4	3.0	3.5	3.0

third order methods (with BD_3 in (3.1.1a))

h	$DQ(G_3)$	$ILM(G_3-BD_3)$	$ML(G_3-BD_2)$	$MML(G_3-BD_3)$
1/10	3.6	2.4	2.9	3.3
	>2.8	>2.3	>2.7	>4.7
1/20	4.5	3.1	3.7	4.7
	>3.9	>2.8	>3.0	>4.1
1/40	5.4	3.9	4.6	6.0

fourth order methods (with BD_4 in (3.1.1a))

h	$DQ(G_4)$	$ILM(G_4-BD_4)$	$ML(G_4-BD_3)$	$MML(G_4-BD_4)$
1/10	4.0	3.2	3.6	3.6
	>3.7	>4.5	>4.0	>3.1
1/20	5.1	4.6	4.8	4.6
	>3.9	>6.2	>4.1	>3.9
1/40	6.3	6.4	6.1	5.7

ACKNOWLEDGEMENT

We would like to thank Joke Blom for programming the numerical experiments.

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APPENDIX

In this appendix we present, successively,

(i) three tables of coefficients of forward differentiation formulas, and of two common LM formulas for ODEs, viz., backward differentiation formulas and Adams-Moulton formulas;

(ii) two lemmas which are needed in:

(iii) proofs of the main results of this paper, as far as they are non-trival (in the opinion of the authors).

Table 1 Coefficients of forward differentiation formulas

$$f'(t_n) \approx \frac{-1}{h} \sum_{\ell=0}^{k} \delta_\ell f(t_{n+\ell}), \quad t_{n+\ell} = t_n + \ell h$$

k	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
1	1	-1				
2	3/2	-2	1/2			
3	11/6	-3	3/2	-1/3		
4	25/12	-4	3	-4/3	1/4	
5	137/60	-5	5	-10/3	5/4	-1/5

Table 2 Coefficients of the backward differentiation formulas

for ODEs $f'(t) = g(t)$: $\sum_{i=0}^{k} a_i f_{n-i} = b_0 g_n$

k	a_0	a_1	a_2	a_3	a_4	a_5	b_0
1	1	-1					1
2	1	-4/3	1/3				2/3
3	1	-18/11	9/11	-2/11			6/11
4	1	-48/25	36/25	-16/25	3/25		12/25
5	1	-300/137	300/137	-200/137	75/137	-12/137	60/137

Table 3 Coefficients of the Adams-Moulton formulas for ODEs $f'(t) = g(t)$: $f_n - f_{n-1} = \sum_{i=0}^{k} b_i g_{n-i}$

k	b_0	b_1	b_2	b_3	b_4	b_5
1	1/2	1/2				
2	5/12	2/3	-1/12			
3	3/8	19/24	-5/24	1/24		
4	251/720	323/360	-11/30	53/360	-19/720	
5	95/288	1427/1440	-133/240	241/720	-137/1440	3/160

LEMMA A.1. *Let* $z_n \geq 0$ *for* $n = 0,1,\ldots,N$, *and suppose that*

$$z_n \leq hC_1 \sum_{i=0}^{n-1} z_i + C_2, \quad n = k,k+1,\ldots,N,$$

where $k > 0$, $h > 0$ *and* $C_i > 0$ $(i=1,2)$. *Suppose, moreover, that* $z_j \leq z/k$ *for* $j = 0,1,\ldots,k-1$. *Then*

$$z_n \leq (hC_1 z + C_2)(1+hC_1)^{n-k}, \quad n = k,k+1,\ldots,N.$$

PROOF. See [4].

LEMMA A.2. *Consider the linear inhomogeneous difference equation with constant coefficients* ζ_j:

$$\text{(A.1)} \qquad \zeta_0 y_{n+k} + \zeta_1 y_{n+k-1} + \ldots + \zeta_k y_n = g_{n+k}, \quad n \geq 0,$$

where $\{g_n\}$ *is a given sequence, independent of the* y_n.
(i) Iff the characteristic polynomial $\zeta(z) := \Sigma_{j=0}^{k} \zeta_j z^{k-j}$ *is simple von Neumann (cf. Section 2.3) then the solution of (A.1) satisfies the inequality*

$$|y_n| \le C\{ \max_{0\le j\le k-1} |y_j| + \sum_{j=k}^{n} |g_j|\}, \quad n \ge k,$$

where C *is independent of* n.

(ii) Iff $\zeta(z)$ *is Schur (cf. Section 2.3) then the solution of (A.1) satisfies the inequality*

$$|y_n| \le C\{ \max_{0\le j\le k-1} |y_j| + \max_{k\le j\le n} |g_j|\}, \quad n \ge k,$$

where C *is independent of* n.

PROOF. See [4].

PROOF OF THEOREM 2.2.1. Taylor expansion of $Y(t_{n+j},t_{n-i})$ around (t_n,t_n) yields

$$\underset{\sim}{L}_n[Y] = \sum_{i=0}^{k} \Big\{\alpha_i \sum_{q=0}^{p} \frac{1}{q!} h^q(-i\frac{\partial}{\partial t}-i\frac{\partial}{\partial s})^q Y(t,s)$$

$$+ \sum_{j=-k}^{k} [\beta_{ij}-\gamma_{ij}h\frac{\partial}{\partial s}] \sum_{q=0}^{p} \frac{1}{q!} h^q(j\frac{\partial}{\partial t}- i\frac{\partial}{\partial s})^q Y(t,s)\Big\}\Big|_{(t_n,t_n)}$$

$$+ \mathcal{O}(h^{p+1}) \text{ as } h \to 0.$$

Writing this formula in the form

$$\underset{\sim}{L}_n[Y] = \sum_{q=0}^{p} \frac{1}{q!} h^q(D_qY)\big|_{(t_n,t_n)} + \mathcal{O}(h^{p+1})$$

and expanding the differential operator D_q by the binomial theorem we find

$$D_q = \sum_{i=0}^{k}\Big\{\alpha_i(-i\frac{\partial}{\partial t}- i\frac{\partial}{\partial s})^q + \sum_{j=-k}^{k} [j\beta_{ij}\frac{\partial}{\partial t} - (i\beta_{ij}+q\gamma_{ij})\frac{\partial}{\partial s}][j\frac{\partial}{\partial t}- i\frac{\partial}{\partial s}]^{q-1}\Big\}$$

$$= \sum_{\ell=0}^{q} \sum_{i=0}^{k}\Big\{(-i)^q\alpha_i - \sum_{j=-k}^{k} j^{q-\ell}(-i)^{\ell-1}[i\beta_{ij}+\ell\gamma_{ij}]\Big\}\binom{q}{\ell}(\frac{\partial}{\partial t})^{q-\ell}(\frac{\partial}{\partial s})^{\ell},$$

where $(-i)^{\ell-1}\ell$ is assumed to be zero for $i = \ell = 0$. Equating to zero all terms in the $\Sigma_{\ell=0}^{q}$ yields the order equations (2.2.3) and at the same time $\underset{\sim}{L}_n(Y) = O(h^{p+1})$ as required in Definition 2.2.1. □

PROOF OF THEOREM 2.2.2. Taylor expansion of $Y(t_{n+j}, t_{n-i})$ around (t_n, t_n) yields

$$Y(t_{n+j}, t_{n-i}) = \sum_{q=0}^{p} \frac{1}{q!} h^q [j\frac{\partial}{\partial t} - i\frac{\partial}{\partial s}]^q Y(t,s)|_{(t_n, t_n)} + \\ + O(h^{p+1}) \text{ as } h \to 0.$$

In order to exploit the fact that $Y(t,t) \equiv 0$ (see definition 2.2.1), we introduce the variables $u = t + s$ and $v = t - s$ and write

$$Y(t,s) = Y(\frac{u+v}{2}, \frac{u-v}{2}) =: Z(u,v).$$

The identity $Y(t,t) \equiv 0$ implies that Z and all its derivatives with respect to u vanish for $u = 2t$ and $v = 0$. In the following we use the notation

$$Z^{(n,m)} := \frac{\partial^n \partial^m Z}{\partial u^n \partial v^m}(2t_n, 0).$$

By means of the binomial theorem we have

$$\text{(A.2)} \qquad Y(t_{n+j}, t_{n-i}) = \sum_{q=0}^{p} \frac{1}{q!} h^q [(j-i)\frac{\partial}{\partial u} + (j+i)\frac{\partial}{\partial v}]^q Z(u,v)|_{(2t_n, 0)} + O(h^{p+1})$$

$$= \sum_{q=0}^{p} \sum_{\ell=0}^{q} \frac{1}{q!} h^q \binom{q}{\ell} (j-i)^{q-\ell} (j+i)^{\ell} Z^{(q-\ell,\ell)} + O(h^{p+1}) \text{ as } h \to 0$$

and

$$\text{(A3)}\quad hY_s(t_{n+j},t_{n-i}) = \sum_{q=0}^{p}\sum_{\ell=0}^{q}\frac{1}{q!}h^{q+1}\binom{q}{\ell}(j-i)^{q-\ell}(j+i)^{\ell}[Z^{(q-\ell+1,\ell)} - Z^{(q-\ell,\ell+1)}] + \mathcal{O}(h^{p+1})$$

$$= \sum_{q=0}^{p}\sum_{\ell=0}^{q}\frac{1}{q!}h^{q}\binom{q}{\ell}(j-i)^{q-\ell-1}(j+i)^{\ell-1}[qj+qi-2\ell j]Z^{(q-\ell,\ell)} + \mathcal{O}(h^{p+1}) \text{ as } h \to 0.$$

Substitution of (A.2) and (A.3) into $\underset{\sim}{L}_n[Y]$ and using $Z^{(q,0)} = 0$ yields

$$\underset{\sim}{L}_n[Y] = \sum_{q=1}^{p}\frac{1}{q!}h^{q}\sum_{\ell=1}^{q}\binom{q}{\ell}B_{q\ell}Z^{(q-\ell,\ell)} + \mathcal{O}(h^{p+1}) \text{ as } h \to 0$$

where $B_{q\ell}$ is defined in (2.2.4). This proves the theorem. □

PROOF OF THEOREM 2.3.1.

PROOF. Taylor expansion in a fixed point $t = t_n$ yields, respectively,

$$y(t_{n-i}) = \sum_{q=0}^{m}\frac{1}{q!}\left(-ih\frac{d}{dt}\right)^{q}y(t_n) + \mathcal{O}(h^{m+1}),$$

$$Y_{n-i}(t_{n+j}) = Y(t_{n+j},t_{n-i}) - E_{n-i}(h;t_{n+j})$$

$$= \sum_{q=0}^{m}\frac{1}{q!}h^{q}\left(j\frac{\partial}{\partial t}-i\frac{\partial}{\partial s}\right)^{q}Y(t_n,t_n) + \mathcal{O}(h^{r}+h^{m+1})$$

$$= \sum_{q=0}^{m}\frac{1}{q!}h^{q}\sum_{\ell=0}^{q}j^{q-\ell}(-i)^{\ell}\binom{q}{\ell}\frac{\partial^{q-\ell}\partial^{\ell}Y}{\partial t^{q-\ell}\partial s^{\ell}}(t_n,t_n) + \mathcal{O}(h^{r}+h^{m+1})$$

$$K_{n-i}(t_{n+j}) = K(t_{n+j},t_{n-i},y(t_{n-i})) = \frac{\partial}{\partial s}Y(t_{n+j},t_{n-i})$$

$$= \sum_{q=0}^{m}\frac{1}{q!}h^{q-1}\sum_{\ell=0}^{q}j^{q-\ell}(-i)^{\ell-1}\binom{q}{\ell}\ell\frac{\partial^{q-\ell}\partial^{\ell}Y}{\partial t^{q-\ell}\partial s^{\ell}}(t_n,t_n) + \mathcal{O}(h^{m}).$$

From these expansions it is immediate that the VLM formula (2.1.4) satisfies the relation

(A.4) $$\sum_{i=0}^{k}\left\{\alpha_i \theta y(t_{n-i}) + \sum_{j=-k}^{k}[\beta_{ij}Y_{n-i}(t_{n+j}) - h\gamma_{ij}K_{n-i}(t_{n+j})]\right\}$$

$$= \sum_{q=0}^{m} h^q\left\{\frac{\theta A_q}{q!}\frac{d^q y}{dt^q}(t_n) + \sum_{\ell=0}^{q}\left(C_{q\ell} - \frac{A_q}{\ell!(q-\ell)!}\right)\frac{\partial}{\partial t}^{q-\ell}\frac{\partial}{\partial s}^{\ell} Y(t_n,t_n)\right\}$$

$$+ \mathcal{O}(h^r + h^{m+1})$$

where A_q and $C_{q\ell}$ are defined by (2.3.2) and (2.2.3), respectively. Under the conditions of the theorem it is easily verified that this equation leads to (2.3.3). Furthermore, (2.3.3) is obviously the m-times differentiated form of equation (1.1). □

PROOF OF THEOREM 2.3.2. Let Y(t,s) be given by (1.6) where y(t) is the exact solution of (1.1), then we may write for $n \geq k$

$$\underset{\sim}{L}_n(Y) \equiv \underset{\sim}{L}_n(Y) - \sum_{i=0}^{k}[\alpha_i y_{n-i} + \sum_{j=-k}^{k}(\beta_{ij}Y_{n-i}(t_{n+j}) - h\gamma_{ij}K_{n-i}(t_{n+j}))]$$

$$= \sum_{i=0}^{k}\left\{\alpha_i \varepsilon_{n-i} + \sum_{j=-k}^{k}[\beta_{ij}(Y(t_{n+j},t_{n-i}) - Y_{n-i}(t_{n+j}))\right.$$

$$\left. - h\gamma_{ij}(K(t_{n+j},t_{n-i},y(t_{n-i})) - K_{n-i}(t_{n+j}))]\right\}.$$

Substitution of the functions Y(t,s) and $Y_n(t)$ and using (2.1.3) and (2.3.6b) leads to

$$\text{(A.5)} \qquad L_n(Y) = \sum_{i=0}^{k} \Big\{ \alpha_i \varepsilon_{n-i} + \sum_{j=-k}^{k} [\beta_{ij}\Big(h \sum_{\ell=0}^{n-i} w_{n-i,\ell} \Delta K(t_{n+j}, t_\ell, y(t_\ell), y_\ell)$$

$$+ E_{n-i}(h; t_{n+j})\Big) - h\gamma_{ij} \Delta K(t_{n+j}, t_{n-i}, y(t_{n-i}), y_{n-i})]\Big\}.$$

Thus, we have found for the errors ε_n the relation

$$\text{(A.6)} \qquad \sum_{i=0}^{k} \alpha_i \varepsilon_{n-i} = v_n, \quad n \geq k^*, \text{ where}$$

$$v_n = L_n(Y) - \sum_{i=0}^{k} \sum_{j=-k}^{k} \Big[h\beta_{ij} \sum_{\ell=0}^{n} w_{n-i,\ell} \Delta K(t_{n+j}, t_\ell, y(t_\ell), y_\ell)$$

$$+ \beta_{ij} E_{n-i}(h; t_{n+j}) - h\gamma_{ij} \Delta K(t_{n+j}, t_{n-i}, y(t_{n-i}), y_{n-i}) \Big].$$

We now proceed with the two cases (a) and (b) separately.

(a) $\alpha(z) \equiv \alpha_0 z^k$, $\alpha_0 \neq 0$.

We want to apply the discrete Gronwall inequality stated in Lemma A.1 in order to derive an upper bound for the solution of this linear difference equation, and therefore we need an upper bound for $|v_n|$. A straightforward calculation yields

$$\text{(A.7)} \qquad |v_n| \leq T(h) + \sum_{i=0}^{k} \sum_{j=-k}^{k} [bwL_1 h \sum_{\ell=0}^{n} |\varepsilon_\ell| + cL_1 h |\varepsilon_{n-i}| + bE(h)]$$

$$\leq C_0 h \sum_{\ell=0}^{n} |\varepsilon_\ell| + C_1 E(h) + T(h),$$

where C_0 and C_1 are constants independent of h and n (in the following all constants C_j will be independent of h and n). From (A.6) it follows that

$$|\alpha_0||\varepsilon_n| \le C_0 h \sum_{\ell=0}^{n} |\varepsilon_\ell| + C_1 E(h) + T(h)$$

so that for h sufficiently small

$$|\varepsilon_n| \le \frac{1}{|\alpha_0| - C_0 h} [C_0 h \sum_{\ell=0}^{n-1} |\varepsilon_\ell| + C_1 E(h) + T(h)]$$

$$\le C_2 h \sum_{\ell=0}^{n-1} |\varepsilon_\ell| + C_3 [E(h) + T(h)].$$

Application of Lemma A.1 (with $z = k^* \delta(h)$) yields

$$|\varepsilon_n| \le (1 + C_2 h)^{n-k^*} (k^* h C_2 \delta(h) + C_3 [E(h) + T(h)]),$$

$$n = k^*, \ldots, N.$$

Since $nh \le T - t_0$, part (a) of the theorem is immediate.

(b) $\alpha(z)$ is simple von Neumann, $\beta(z) \equiv 0$.

Instead of directly applying Lemma A.1 to the inequality (obtained from (A.6))

$$\sum_{i=0}^{k} |\alpha_i||\varepsilon_{n-i}| \le |v_n|,$$

we first apply Lemma A.2 (i) to obtain the "sharper" inequality

(A.8) $$|\varepsilon_n| \le C_0 [\delta(h) + \sum_{j=k}^{n} |v_j|], \quad n \ge k^*.$$

Unfortunately, if we use the upper bound (A.7) for $|v_j|$ and then apply Lemma A.1, we cannot prove convergence. However, by using the property $\beta(z) \equiv 0$, that is $\beta_i = \Sigma_{j=-k}^{k} \beta_{ij} = 0$, a sharper upper bound than (A.7) can be derived. To that end we write

$$\left|\sum_{j=-k}^{k} \beta_{ij} \Delta K(t_{n+j}, t_\ell, y(t_\ell), y_\ell)\right| = \sum_{j=-k}^{k} \beta_{ij} [\Delta K(t_n, t_\ell, y(t_\ell), y_\ell)$$

$$+ \Delta K(t_{n+j}, t_\ell, y(t_\ell), y_\ell) - \Delta K(t_n, t_\ell, y(t_\ell), y_\ell)]|$$

$$\leq bLh \sum_{j=-k}^{k} |j\varepsilon_\ell|,$$

and, similarly,

$$\left|\sum_{j=-k}^{k} \beta_{ij} E_{n-i}(h; t_{n+j})\right| \leq b \sum_{j=-k}^{k} \Delta E(h).$$

In this way we obtain instead of (A.7) the upper bound

$$\text{(A.9)} \qquad |v_n| \leq T_n(h) + \sum_{i=0}^{k} \sum_{j=-k}^{k} [bwL|j|h^2 \sum_{\ell=0}^{n} |\varepsilon_\ell| + cL_1 h|\varepsilon_{n-i}| + b\Delta E(h)]$$

$$\leq C_1 h \sum_{i=0}^{k} [|\varepsilon_{n-i}| + h \sum_{\ell=0}^{n} |\varepsilon_\ell|] + C_2 \Delta E(h) + T(h).$$

Substitution into (A.8) yields the inequality

$$|\varepsilon_n| \leq C_3 \left\{\delta(h) + h \sum_{j=k}^{n} \left[\sum_{i=0}^{k} |\varepsilon_{j-i}| + h \sum_{\ell=0}^{j} |\varepsilon_\ell| + h^{-1}\Delta E(h) + h^{-1} T(h)\right]\right\}.$$

It is easily verified that

$$\sum_{j=k}^{n} \sum_{i=0}^{k} |\varepsilon_{j-i}| \leq (k+1) \sum_{j=0}^{n} |\varepsilon_j|.$$

Hence,

$$|\varepsilon_n| \leq C_4 \left\{\delta(h) + h\left[(1+nh) \sum_{\ell=0}^{n} |\varepsilon_\ell| + nh^{-1}\Delta E(h) + nh^{-1} T(h)\right]\right\}.$$

Since $nh \le T - t_0$ we find for h sufficiently small

$$|\varepsilon_n| \le C_5 h \sum_{\ell=0}^{n-1} |\varepsilon_\ell| + C_6 h^{-1}[h\delta(h) + \Delta E(h) + T(h)].$$

Finally, by applying Lemma A.1 we arrive at the estimate

$$|\varepsilon_n| \le (1+C_5 h)^{n-k^*}\Big(k^* h C_5 \delta(h) + C_6 h^{-1}[h\delta(h)+\Delta E(h)+T(h)]\Big),$$

from which part (b) of the theorem follows. □

PROOF OF THEOREM 2.3.4. Following the first lines of the proof of Theorem 2.3.2 we obtain the following relation, analogous to (A.5), where $K_{rs} := K(t_r, t_s)$

(A10) $$\sum_{i=0}^{k} \sum_{j=-k}^{k} \gamma_{ij} K_{n+j,n-i}\varepsilon_{n-i} = \sum_{i=0}^{k} \sum_{j=-k}^{k} \beta_{ij}\Big[\sum_{\ell=0}^{n} w_{n-i,\ell} K_{n+j,\ell}\varepsilon_j + h^{-1} E_{n-i}(h;t_{n+j})\Big] - h^{-1}\underset{\sim}{L}_n(Y), \quad n \ge k^*.$$

Now we write $K_{n+j,n-i} = K_{nn} + (K_{n+j,n-i} - K_{nn})$ and $K_{n+j,\ell} = K_{n\ell} + (K_{n+j,\ell} - K_{n\ell})$ and rewrite (A.10) to obtain

(A.11) $$\sum_{i=0}^{k} \gamma_i \varepsilon_{n-i} = v_n, \quad n \ge k^*,$$

where

$$K_{nn} v_n = h \sum_{i,j} \gamma_{ij} \Big(\frac{K_{nn} - K_{n+j,n-i}}{h}\Big)\varepsilon_{n-i} + \sum_{i,j} \beta_{ij} \sum_{\ell} w_{n-i,\ell} K_{n\ell}\varepsilon_\ell +$$

$$+ h \sum_{i,j} \beta_{ij} \sum_{\ell} w_{n-i,\ell} \Big(\frac{K_{n+j,\ell} - K_{n\ell}}{h}\Big)\varepsilon_\ell +$$

$$+ h^{-1} \sum_{i,j} \beta_{ij} E_{n-i}(h;t_{n+j}) - h^{-1}\underset{\sim}{L}_n(Y).$$

Since $\gamma(z)$ is Schur, we may apply Lemma A.2 (ii) to (A.11) and find

$$|\varepsilon_n| \le C\{\delta(h) + \max_{k\le j\le n} |v_j|\}, \quad n \ge k^*. \tag{A.12}$$

where C (and all subsequent C_i) is independent of h and n. So we have to find bounds on $|v_j|$. Using the conditions of the theorem, we find

$$|v_r| \le C_1 h \sum_{i,j} |\gamma_{ij}|(j+i)|\varepsilon_{r-i}| + \left|\sum_{i=0}^{k} \beta_i \sum_{\ell=0}^{r} w_{r-i,\ell} K_{r\ell}\varepsilon_\ell\right|$$

$$+ C_2 hw \sum_{i,j} j|\beta_{ij}| \sum_{\ell=0}^{r} |\varepsilon_\ell| + h^{-1}\left|\sum_{i,j} \beta_{ij} E_{r-i}(h;t_{r+j})\right| + h^{-1}|\underset{\sim}{L}_r(Y)|,$$

$$r \ge k^*.$$

Now we use the condition $\beta(z) \equiv 0$, i.e., $\beta_i = 0$, and (2.3.6a) to obtain (cf. the derivation of (A.9) in the proof of Theorem 2.3.2)

$$|v_r| \le C_3\left\{h \sum_{i=0}^{k} |\varepsilon_{r-i}| + h \sum_{\ell=0}^{r} |\varepsilon_\ell| + h^{-1}\Delta E(h)\right\} + h^{-1}T(h), \quad r \ge k^*,$$

$$\le C_4\left\{h \sum_{\ell=0}^{r} |\varepsilon_\ell| + h^{-1}\Delta E(h)\right\} + h^{-1}T(h).$$

Substituting this into (A.12) we find, for h sufficiently small,

$$|\varepsilon_n| \le C_5\left\{\delta(h) + h^{-1}\Delta E(h) + h^{-1}T(h) + h \sum_{\ell=0}^{n-1} |\varepsilon_\ell|\right\}$$

and application of Lemma A.1 yields the result of the theorem. □

PROOF OF THEOREM 3.3.1. Proceeding as in the proof of Theorem 2.3.2 we derive the relations

$$(A.13)\quad \begin{aligned} \underset{\sim}{L}_n^*[y] &= \sum_{i=0}^{k} [\alpha_i^* \varepsilon_{n-i} - h\gamma_i^* \, \Delta f_{n-i}], \\ \underset{\sim}{L}_n[y] &= \sum_{i=0}^{k} \Big\{ \alpha_i \eta_{n-i} + \sum_{j=-k}^{k} \Big[\beta_{ij} (h \sum_{\ell=0}^{n-i} w_{n-i,\ell} \Delta K(t_{n+j}, t_\ell, y(t_\ell), y_\ell) \\ &\quad + E_{n-i}(h; t_{n+j})) - h\gamma_{ij} \, \Delta K(t_{n+j}, t_{n-i}, y(t_{n-i}), y_{n-i}) \Big] \Big\}. \end{aligned}$$

The first relation is written as (cf. (A.6))

$$(A.14)\qquad \sum_{i=0}^{k} \alpha_i^* \varepsilon_{n-i} = v_n^*,$$

where v_n^* satisfies the inequality (using (1.3') and (1.3"))

$$\begin{aligned} |v_n^*| &:= |\underset{\sim}{L}_n^*[y] + h \sum_{i=0}^{k} \gamma_i^* \Delta f_{n-i}| \\ &\le T_n^*(h) + h \sum_{i=0}^{k} |\gamma_i^*| \, [L_1 |\varepsilon_{n-i}| + L_2 |\eta_{n-i}|]. \end{aligned}$$

Application of Lemma A.2 (i) yields (because $\alpha^*(z)$ is simple von Neumann)

$$(A.15)\qquad |\varepsilon_n| \le C_0 \Big[h \sum_{j=0}^{n} [\, |\varepsilon_j| + |\eta_j| \,] + \delta(h) + \sum_{j=k}^{n} T_j^*(h) \Big]$$

where C_0 is some constant independent of n and h.

For η_n we derive from the second relation in (A.13)

$$(A.16)\qquad \sum_{i=0}^{k} \alpha_i \eta_{n-i} = v_n$$

where v_n is defined as in (A.6).

(a) In the case where $\alpha(z) = \alpha_0 z^k$ we have from (A.7):

$$|\eta_n| \le C_1 [E_n(h) + h \sum_{\ell=0}^{n} |\varepsilon_\ell| + T_n(h)], \quad n \ge k^*$$

for some constant C_1. Substitution into (A.15) yields

$$|\varepsilon_n| \le C_2\Big\{h \sum_{j=k}^{n} [\,|\varepsilon_j| + h \sum_{\ell=0}^{j} |\varepsilon_\ell| + E_j(h) + T_j(h) + h^{-1}T_j^*(h)] + \delta(h) + h\delta^*(h)\Big\}$$

$$\le C_3\Big\{h \sum_{j=0}^{n} |\varepsilon_j| + E_n(h) + T_n(h) + h^{-1}T_n^*(h) + \delta(h) + h\delta^*(h)\Big\}$$

where we have used that $nh \le T - t_0$. From Lemma A.1, part (a) of the theorem easily follows.

(b) Since $\alpha(z)$ is simple von Neumann, we apply Lemma A.2 (i) to (A.16) and use (A.9) (since $\beta(z) \equiv 0$) to find

$$|\eta_n| \le C_4\Big\{\delta^*(h) + \sum_{j=k}^{n} \Big[\sum_{i=0}^{k} (h|\varepsilon_{j-i}| + h^2 \sum_{\ell=0}^{j} |\varepsilon_\ell|) + \Delta E_j(h) + T_j(h)\Big]\Big\}$$

$$\le C_5\Big\{h \sum_{j=0}^{n} |\varepsilon_j| + \delta^*(h) + h^{-1}\Delta E_n(h) + h^{-1}T_n(h)\Big\}.$$

Substitution into (A.15) and applying Lemma A.1 leads to part (b) of the theorem. □